Second-order symmetric Lorentzian manifolds II: structure and global properties

O F Blanco[1], **M Sánchez**[1] **and J M M Senovilla**[2]

[1] Departamento de Geometría y Topología , Facultad de Ciencias, Universidad de Granada Campus Fuentenueva s/n, 18071 Granada, Spain
[2] Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

E-mail: `oihane@ugr.es, sanchezm@ugr.es, josemm.senovilla@ehu.es`

Abstract. We give a summary of recent results on the explicit local form of the second-order symmetric Lorentzian manifolds in arbitrary dimension, and its global version. These spacetimes turn out to be essentially a specific subclass of plane waves.

1. Introduction

Last year we presented the local explicit form of the *second-order symmetric* (or *2nd symmetric*[1] for short) spacetimes in four dimensions [3]. The techniques used for that purpose were specific of the dimension. Our aim now is to introduce new techniques that generalize the result to arbitrary dimension. Recall that 2nd symmetric spacetimes are characterized by the condition

$$\nabla_\rho \nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0.$$

They were introduced by one of the authors in [17] as the simplest natural generalization of the *locally symmetric* spacetimes (those satisfying $\nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0$). In the Riemannian [2],[7],[15] and Lorentzian [6] cases, the locally symmetric spaces are well known. However, 2nd symmetric spaces have been hardly considered in the literature, probably because the local de Rham decomposition implies that 2nd symmetry and local symmetry are equivalent in the proper Riemannian case [16],[22]. In the Lorentzian case such decomposition fails [23], and so does the equivalence. Thus, even though in arbitrary signature *semi-symmetric* spaces (those satisfying $\nabla_{[\rho}\nabla_{\sigma]}R^\alpha{}_{\beta\lambda\mu} = 0$, see for example [20],[21] for the Riemannian case and [12], [14] for the Lorentzian one) have always been regarded as the simple natural generalization of the locally symmetric ones, in indefinite signatures the 2nd symmetry is a simpler natural choice.

In this contribution we announce the following classification of *proper 2nd symmetric* spacetimes (here proper means that they are not locally symmetric, that is $\nabla_\sigma R^\alpha{}_{\beta\lambda\mu} \neq 0$) in arbitrary dimension [4]:

Teorema 1.1 *A n-dimensional proper 2nd symmetric spacetime (M, g) is locally isometric to a product spacetime $(M_1 \times M_2, g_1 \oplus g_2)$ where (M_2, g_2) is a non-flat Riemannian symmetric space*

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[1] The notion of *3-symmetric space* (without ordinal), which is widely spread in the literature, was introduced by Gray [13] in a different context.

and (M_1, g_1) is a generalized Cahen-Wallach space of order 2, defined as $M_1 = \mathbb{R}^{d+2}$ $(d \geq 1)$ endowed with the metric

$$g_1 = -2du\left(dv + p_{ij}(u)x^i x^j du\right) + \delta_{ij}dx^i dx^j, \quad i, j = 2, \ldots, d+1$$

where $\{u, v, x^i\}$ are the Cartesian coordinates of $\mathbb{R}^{d+2}$, each function p_{ij} is affine, i.e., $p_{ij}(u) = \alpha_{ij}u + \beta_{ij}$ for some $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, and $\sum_{i,j=2}^{d+1}(\alpha_{ij})^2 \neq 0$.

If in addition (M, g) is geodesically complete and simply connected, then (M, g) is globally isometric to one of such products.

2. Sketch of the proof of theorem 1.1

As in the four-dimensional case, the starting point for the proof is [17, Theorem 4.2], which states when, in the Lorentzian case, the 2nd symmetry condition does not imply local symmetry. This can only happen when the spacetime possesses a null (non-vanishing) covariantly constant vector field. Therefore, it is locally isometric to a Brinkmann spacetime, whose metric can be written in an appropiate *Brinkmann chart* $\{u, v, x^i\}$ as [5],[24]:

$$ds^2 = -2du(dv + Hdu + W_idx^i) + g_{ij}dx^i dx^j, \quad i, j \in \{2, \ldots, n-1\},$$

where H, W_i and $g_{ij} = g_{ji}$ are functions independent of v, otherwise arbitrary. With this data at hand, the steps of the proof are the following:

(i) A suitable choice of vector field basis simplifies the computation of the equations of 2nd symmetry.

(ii) As a first consequence, the leaves of the foliation $\overline{\mathcal{M}}$ generated by the submanifolds with constant values of u and v are locally symmetric Riemannian manifolds for any Brinkmann chart.

(iii) Using the integrability equations and (ii), one proves that the only non-vanishing components of the curvature in the basis (i) are $\nabla_0 R^1{}_{i0j}$.

(iv) Applying a generalization of the Eisenhart theorem [11], "the spatial part" g_{ij} is splits into two parts, one of them is Ricci-flat which, on using a result in [1], becomes actually flat.

(v) After lengthy computations, one can prove that there exists a Brinkmann chart such that the metric becomes:

$$
\begin{aligned}
ds^2 \;=\; & -2du\left(dv + \left(H_1(u, x^a) + H_2(u, x^{a'})\right)du + W_a(u, x^b)dx^a + W_{a'}(u, x^{b'})dx^{a'}\right) \\
& + \; \delta_{ab}dx^a dx^b + g_{a'b'}dx^{a'}dx^{b'},
\end{aligned}
$$

with $a, b \in \{2, \ldots, d+1\}$, $a', b' \in \{d+2, \ldots, n-1\}$, $g_{a'b'} \neq \delta_{a'b'}$.

(vi) At this stage, the equations can be reorganized in two blocks:

- one corresponding to the equations of local symmetry, whose solutions are already known [6] (see also [2]),
- the other one corresponding to the equations of 2nd symmetry for a plane wave, which are easily solvable.

To develop the proof, the introduction of some mathematical tools is required, namely:

(1) Some operators defined on the tensor bundle $T_s^r\overline{\mathcal{M}}$, denoted as D_0 (transverse operator) and $\overline{\nabla}$ (which arises from the covariant derivative on the leaves), and an adaptation of the exterior derivative to $\overline{\mathcal{M}}$.

(2) The curvature tensor $\overline{R}^s{}_{ijk}$ associated intrinsically to the foliation (i.e., to $\overline{\nabla}$), as well as its naturally associated Ricci tensor $\overline{R}_{ij}$ and scalar curvature $\overline{R}$.

(3) Some tensor fields on $\overline{\mathcal{M}}$ defined as projections of $R^\alpha{}_{\beta\lambda\mu}$ and $\nabla_\rho R^\alpha{}_{\beta\lambda\mu}$, as well as some algebraic lemmas on vector spaces with positive definite inner product.

(4) An appropriate adaptation of the Eisenhart theorem to Brinkmann spacetimes, as mentioned above. This involves the operators D_0 and $\overline{\nabla}$ and ensures the reducibility of the entire family of metrics g_{ij} —which depend on u— simultaneously.

Before dealing with the global part of the proof, we define

- *proper r-th symmetric* spacetimes as those satisfying $\nabla_{\rho_1} \ldots \nabla_{\rho_r} R^\alpha{}_{\beta\lambda\mu} = 0$ but $\nabla_{\rho_1} \ldots \nabla_{\rho_{r-1}} R^\alpha{}_{\beta\lambda\mu} \neq 0$,

- the generalized *Cahen-Wallach spacetimes of order r*, denoted by $CW_r^d(A)$, as the spacetimes $(\mathbb{R}^d, g_A)$, $d \geq 3$ with metric

$$g_A = -2du \left(dv + A_{ij} x^i x^j du\right) + \delta_{ij} dx^i dx^j,$$

where $A = (A_{ij})$ is a square symmetric matrix of dimension $d - 2$ such that $A \equiv A(u) = A^{(r-1)} u^{r-1} + \ldots + A^{(1)} u + A^{(0)}, A^{(r-1)} \neq 0$, and each $A^{(l)}$ is a square symmetric matrix of constants, $l = 1, \ldots, r - 1$. Observe that these spacetimes are proper rth symmetric [17], as well as analytic and geodesically complete [8].

Then, the global result follows directly from:

Proposition 2.1 *For any matrix A as above:*

- *the direct product of $CW_r^n(A)$ and a finite number of Riemannian symmetric (non-necessarily simply connected) spaces is a proper r-th symmetric Lorentzian manifold, which is in fact geodesically complete and analytic, and*

- *if a complete Lorentzian manifold (M, g) is locally isometric to spaces of the form given above, then its universal covering is also of this type.*

Remark 2.2 The techniques used to solve the arbitrary dimensional case differ from the techniques used in four dimensions since in the latter case the Petrov classification of the Weyl tensor [19] was used. Such classification is not available in arbitrary dimension with the same simplicity, though some generalizations have already apeared in the literature (see [9],[10],[18]). Therefore, we had to refocuse the problem in arbitrary dimension using a different, and more powerful, viewpoint.

References
[1] Alekseevskii DV and Kimelfeld BN 1975 Structure of homogeneous Riemannian spaces with zero Ricci curvature (Russian) *Funkcional. Anal. i Prilozen.* **9** 2 5-11
[2] Besse A L *Einstein manifolds* (Springer, 1987)
[3] Blanco O F, Sánchez M and Senovilla J M M 2010 Complete classification of second-order symmetric spacetimes *J. Phys.: Conf. Ser.* **229** 012021
[4] Blanco O F, Sánchez M and Senovilla J M M Structure of second-order symmetric Lorentzian manifolds, in preparation
[5] Brinkmann H W 1925 Einstein spaces which are mapped conformally on each other *Math. Ann.* **94** 119-45
[6] Cahen M, Wallach F 1970, Lorentzian symmetric spaces *Bull. Amer. Math. Soc.* **76** 585-591
[7] Cartan É 1926 Sur une classe remarquable d'espaces de Riemann *Bull. Soc. Math. France* **54** 214-64
Cartan É 1927 Sur une classe remarquable d'espaces de Riemann *Bull. Soc. Math. France* **55** 114-34

[8] Candela A M, Flores J L and Sánchez M 2003 On general plane fronted waves. Geodesics *Gen. Relat. Gravit.* **35** 631-649.

[9] Coley A. 2008 Classification of the Weyl Tensor in Higher Dimensions and Applications *Class.Quant.Grav.* **25** 033001

[10] Coley A. Milson R. Pravda V. and Pravdov Ì A. 2004 Classification of the Weyl Tensor in Higher Dimensions *Class.Quant.Grav.* **21** L35-L42

[11] Eisenhart L P 1923, Symmetric tensors of the second order whose first covariant derivatives are zero *Trans. Amer. Math. Society* **25** 2 297-306

[12] Eriksson I and Senovilla J M M 2010 Note on (conformally) semi-symmetric spacetimes *Class. Quantum Grav.* **27** 027001

[13] Gray A 1972 Riemannian manifolds with geodesic symmetries of order 3 *J. Differential Geometry* **7**, 343-369

[14] Haesen S and Verstraelen L 2004 Classification of the pseudo-symmetric space-times, *J. Math. Phys.* **45** 2343-6

[15] Helgason S *Differential Geometry, Lie Groups, and Symmetric Spaces* (Academic Press, New York, 1978)

[16] Nomizu K and Ozeki H 1962 A theorem on curvature tensor fields *Proc. Nat. Acad. Sci. USA* **48 206-7**

[17] Senovilla J M M 2008 2nd-order symmetric Lorentzian manifolds I: characterization and general results, *Class. Quantum Grav.* **25** 245011

[18] Senovilla J M M 2010 Algebraic classification of the Weyl tensor in higher dimensions based on its 'superenergy' tensor *Class. Quantum Grav.* **27** 222001

[19] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambrigde University Press., Cambrigde, 2003)

[20] Szabó Z I 1982 Structure theorems on Riemannian spaces satisfying $R(X,Y)\dot{R} = 0$: I. The local version *J. Differ. Geom.* **17** 531-82

[21] Szabó Z I 1985 Structure theorems on Riemannian spaces satisfying $R(X,Y)\dot{R} = 0$: I. The global versions *Geom. Dedicata* **19** 65-108

[22] Tanno S 1972 Curvature tensors and covariant derivatives *Ann. Mat. Pura Appl.* **96** 233-41

[23] Wu H 1967 Holonomy groups of indefinite metrics *Pacific J. Math.* **20** 351-92

[24] Zakharov V D 1973 *Gravitational Waves in Einstein's theory* (New York: Wiley)